We have not attempted to recover from here the solution given by Professors Powers and Tapley, for there is enough information given in their note and in this comment to carry out this exercise to a successful conclusion.

Of course Delaunay's transformation is restricted to elliptic Keplerian motions. There are known universal Delaunay's variables, but we have not succeeded yet in producing for them a set of convenient partial derivatives. Hence we do not know yet how to produce the solution of the coast-arc in universal variables along the lines we have taken here for elliptic motions.

References

[1] Powers, W. F. and Tapley, B. D., "Canonical Transformation Applications to Optimal Trajectory Analysis," *AIAA Journal*, Vol. 7, No. 3, March 1969, pp. 394–399.

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Reply by Authors to A. Deprit

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THE solution to the elliptic coast-arc problem that Dr. Deprit has developed is indeed an interesting way to attack the problem. The purpose of this Reply is to point out some of the similarities and differences of the two solutions.

First, the solution of Ref. 1 was developed in terms of Poincare variables because, as is well known, they are well defined at circular conditions, whereas the Delaunay variables involve the argument of perigee that is undefined at circular conditions. Since many of the space guidance problems of interest involve circular or near-circular conditions at one point or another, it was imperative that we obtain a solution that is valid for such cases.

Second, Eqs. (35) and (36) of Ref. 1 are actually transformation equations between the Poincare variables and the set $\{\alpha_1, \ldots, \alpha_5, \beta_1, \ldots, \beta_5\}$, which represents a full set of constant parameters for the coast-arc. One can use Eqs. (23) and (A1) to obtain the transformation equations between the polar variables and the coast-arc parameters $\{\alpha,\beta\}$. The resultant set of equations would then be analogous to the list of transformations in Dr. Deprit's paper. In the $\{\alpha,\beta\}$ system, the variational Hamiltonian on the coast-arc is $K \equiv 0$, whereas $K = (\mu^2/L^3)\lambda_l$ on the coast-arc in Dr. Deprit's solution. The essential fact is that even though both resultant Hamiltonians have simple forms, the transformation equations between the polar system and either orbital parameter system still are cumbersome. However, if one need not transform back to the polar system, then both systems have desirable properties.

Third, the solution of Ref. 1 may be extended easily to take into account nonplanar angles and multipliers, e.g., i and Ω. For example, if $q_6 \equiv i$, $q_7 \equiv \Omega$, $p_6 \equiv \Lambda_i$, $p_7 \equiv \Lambda_\Omega$, then $(dq_6/d\theta) = 0$ and $(dq_7/d\theta) = 0$ on the coast-arc and the Hamiltonian for the coast-arc [i.e., Eq. (29)] is unchanged. Thus, it may be shown[2] that if the S-function of Eq. (34) is denoted by S^*, then $S = S^* + \alpha_6 q_6 + \alpha_7 q_7$ is a complete solution of the new Hamilton-Jacobi equation for the coast arc.

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Comments on "Study of Nonlinear Systems"

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IN recent Technical Notes B. V. Dasarathy and P. Srinivasan have outlined methods of solving certain types of nonlinear differential systems of third- and second-order by use of transformation functions. In their paper, it is shown that use of nonlinear transformation function actually reduces the nonlinear system into an equivalent linear system, which can easily be solved to obtain the response of the original system. The transformations indicated in their Refs. 1 and 2 is the transformation of both independent and dependent variables. This transformation technique is applied to solve the nonlinear system of the type given below

$$Md^2x/dt^2 + Cx^n(dx/dt) + K_1x^n + K_2x^{2n+1} = 0$$

where M, C, K_1 and K_2 are constant parameters.

When this is applied the previous equation is transformed into a linear system and thus gives an implicit relation between independent and dependent variables in terms of a common parameter.

A study of the nonlinear equation chosen by the aforementioned authors, revealed a certain relationship between the coefficient of dx/dt and the other functions of x in the equation. As a result of this property, the equation can be integrated. This Note presents two approaches to solve the nonlinear system, by use of this property.

Ist Approach

The equation to be solved is

$$M(d^2x/dt^2) + Cx^n(dx/dt) + K_1x^n + K_2x^{2n+1} = 0 \quad (1)$$

This can be written as

$$(d^2x/dt^2) + f(x)(dx/dt) + F(x) = 0 \quad (2)$$

where

$$f(x) = (C/M)x^n$$

$$F(x) = (1/M)(K_1x^n + K_2x^{2n+1})$$

By change of dependent variable to independent variable, we get

$$(d^2t/dx^2) = f(x)(dt/dx)^2 + F(x)(dt/dx)^3$$

Since the transformed equation has dependent variable as t and this does not appear in the equation as such the following substitution can be made $R = dt/dx$, and we have

$$dR/dx = f(x)R^2 + F(x)R^3 \quad (3)$$

Equation (3) is of "Ablesche" type. By close observation of the coefficient of R^2 and R^3, it is seen that they satisfy the following relation, viz.,

$$(d/dx)[F(x)/f(x)] = \lambda f(x)$$

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where λ is a constant.

On account of this property, Eq. (3) can be easily reduced to integrable form by

$$R = [f(x)/F(x)]\cdot u(x)$$

Then we get

$$du/dx = (f^2/F)(u^3 + u^2 + \lambda u)$$

Here the variables are separable. Hence, we get

$$u/(u^2 + u + \lambda)^{1/2}\{[u + \tfrac{1}{2} - (\tfrac{1}{4} - \lambda)^{1/2}]/[u + \tfrac{1}{2} + (\tfrac{1}{4} - \lambda)^{1/2}]\}^{\delta} = k(k_1 + k_2x^{n+1})$$

where

$$\delta = 1/4(\tfrac{1}{4} - \lambda)^{1/2}$$

From this if we can express u as a function of x, say $V(x)$, we get

$$t = \int \frac{Cx^n}{K_1x^n + K_2x^{2n+1}}\cdot V(x)dx + \text{const}$$

By application of this method to equations of surge tank, i.e.,

$$\ddot{x} + bx\dot{x} + ax = 0$$

we get

$$dR/dx = bxR^2 + axR^3$$

where $R = dt/dx$. Here already variables are separable. Hence by integration we get

$$dR/(bR^2 + aR^3) = xdx$$

$$\therefore a/b^2 \log(b/R + a) - 1/bR = x^2/2 + \text{const}$$

From this if it is possible to express R as a function of x, then one more integration will enable us to get an independent variable.

2nd Approach

Since the independent variable is not present we can write the given equation as

$$PdP/dx + f(x)P + F(x) = 0 \tag{4}$$

where $P = dx/dt$. This is again Ablesche type. Now through the change of variable, by $\xi = \int f(x)dx$, and because of relation in $f(x)$ and $F(x)$, we get

$$dP/d\xi = -[P + K_1/C + (K_2/C^2)\cdot M(n + 1)\cdot \xi]/P$$

This equation can be easily reduced to integrable form by changing the variable by

$$R = K_1/C + K_2/C^2\cdot M(n + 1)\cdot \xi$$

Then we get

$$dP/dR = -(1/\mu)(P + R/P)$$

where

$$\mu = K_2(n + 1)M/C^2$$

This equation can be very easily solved, being a homogeneous form.

The previous method can be applied to the equation of Ref. 2 viz.,

$$\dddot{x} - (\dot{x}/x)\ddot{x} + bx^2\dot{x} = 0$$

By writing $P = dx/dt$, we get

$$P^2P'' + PP'^2 - P^2P'/x + bx^2P = 0 \tag{5}$$

where

$$P' = dP/dx$$

let

$$\xi = \int xdx$$

then we get $P\ddot{P} + \dot{P}^2 + b = 0$, where $\dot{P} = dP/d\xi$. Integrating this equation we get

$$(dx/dt)^2 = P^2 = 2k\xi - b\xi^2 + K_1$$

$$\therefore t = \int dx \bigg/ \left(Kx^2 - \frac{b}{4}\cdot x^4 + K_1\right)^{1/2} + \text{const}$$

where K_1 and K are the constants of integration.

Of course this equation can be integrated without any transformation or change of variable, as this can be written as

$$(d/dt)(\ddot{x}/x) + (b/2)(d/dt)(x^2) = 0$$

By integration we have

$$\ddot{x}/x + b/x^2 2x = K$$

by again integrating we get

$$t = \int \frac{dx}{[Kx^2 - (b/4)\cdot x^4 + K_1]^{1/2}} + \text{const}$$

where K_1 and K are the constants of integration.

Further, it is shown that systems with changed coefficient namely of the type

$$(d^2x/dt^2) + F(x)(dx/dt) + f(x) = 0 \tag{6}$$

where

$$F(x) = (K_1/M)x^n + (K_2/M)x^{2n+1}$$

$$f(x) = (C/M)x^n$$

can also be reduced to an integrable form. This is achieved by first converting this system to standard Riccati type of equation which is then easily reduced to linear system by change of variable for final solution.

Method of Solution

$$(d^2x/dt^2) + F(x)(dx/dt) + f(x) = 0$$

This can be written as

$$P(dP/dx) + F(x)P + f(x) = 0 \tag{7}$$

(Ablesche type of equations) where $P = dx/dt$. This is then transformed by $\xi = \int f(x)dx$, and by substituting

$$P = dx/dt = u - A\xi - (B/2)\xi^2$$

where A and B are constant, given by $A = K_1/C$, $B = [K_2(n + 1)M]/C^2$. We get the following equations:

$$d\xi/du = [A\xi + (B/2)\xi^2 - u] \tag{8}$$

This is a Riccati differential equation in ξ which can be easily converted to linear differential equation of 2nd order namely

$$d^2R/du^2 - AdR/du - \lambda Ru = 0 \tag{9}$$

where

$$\xi = -(1/\lambda R)\cdot dR/du \text{ and } \lambda = B/2$$

This linear equation with variable coefficient can be easily transformed to Airy's differential equation. Hence it can be solved in closed form. The solution is given by

$$R = \frac{M^{1/2}}{\lambda} \exp\left(\frac{Au}{2}\right)\left[C_1I_{1/3}\left(\frac{2}{3\lambda^{4/3}}\cdot M^{2/3}\right) + C_2K_{1/3}\left(\frac{2}{3\lambda^{4/3}}\cdot M^{2/3}\right)\right] = f(u)$$

where C_1 and C_2 are the constant of integrations, and $M = A^2/4 + \lambda u$. Hence, we get

$$\xi = -f'(u)/\lambda f(u) \tag{10}$$

If it is possible to express from Eq. (10) u as function of ξ then from

$$dx/dt = u - A\xi - (B/2)\xi^2$$

we will be able to get the independent variable by one integration.

Conclusion

Hence it is seen that the nonlinear equations of the type

$$d^2x/dt^2 + f(x)(dx/dt) + F(x) = 0$$

or

$$d^2x/dt^2 + F(x)(dx/dt) + f(x) = 0$$

when $f(x)$ and $F(x)$ satisfy the following relationship, namely

$$(d/dx)[F(x)/f(x)] = \lambda f(x)$$

where λ is a constant, can be reduced to an integrable form.

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